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2001 J. Phys. A: Math. Gen. 34 8151

(http://iopscience.iop.org/0305-4470/34/39/314)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 8151–8169 PII: S0305-4470(01)24743-X

Correlation functions of the higher spin *XXX* chains

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Received 8 May 2001, in final form 1 August 2001
Published 21 September 2001
Online at stacks.iop.org/JPhysA/34/8151

Abstract

Using the algebraic Bethe ansatz, we consider the correlation functions of the integrable higher spin chains. We apply a method recently developed for the spin $\frac{1}{2}$ Heisenberg chain, based on the solution of the quantum inverse problem. We construct a representation for the correlation functions on a finite chain for arbitrary spin. Then we show how the string solutions of the Bethe equations can be considered in the framework of this approach in the thermodynamic limit. Finally, a multiple integral representation for the spin 1 zero-temperature correlation functions is obtained in the thermodynamic limit.

PACS numbers: 71.45Gm, 02.30.Ik, 03.65.Fd, 11.30.Na, 75.10.Jm

1. Introduction

A new method of computation of the correlation functions and of the form factors of the Heisenberg spin $\frac{1}{2}$ chains developed in [1–3] based on the algebraic Bethe ansatz [4] and the resolution of the quantum inverse problem [1, 5] has provided a possibility to calculate a very large class of correlation functions for a finite chain and in the thermodynamic limit. The zero temperature correlation functions (which are defined as mean values of some products of local operators with respect to the ground state of the model) were obtained [3] as multiple integrals, which coincide for the case without magnetic field with the results obtained using the vertex operators approach [6, 7]. However, the new method gave a better understanding of the structure of these results. It was shown, in particular, that the expressions under integrals can be separated into two parts with different origin: an algebraic part which depends on the choice of local operators and does not depend on the choice of the ground state and an analytic (or determinant) part which, in contrast, is fixed uniquely by the ground state.

This very particular structure permits to hope that these results can be generalized for several more general situations in particular for the temperature-dependent correlation functions for the Heisenberg spin $\frac{1}{2}$ but also for other integrable models with the same

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R-matrix, and first of all for the higher spin chains (they can be considered as the first step to the future generalizations). These two apparently very different problems have however one common detail: the main difficulty is the analysis of excited states in the first case or of more complicated ground states for the second one. In both cases we should deal with *bound states* or, more precisely, with the string solutions of the Bethe equations. The understanding of the influence of these bound states (quantum breathers) is a very important step in the calculation of the temperature- and time-dependent correlation functions.

For this reason, before considering a more complicated example of temperature-dependent correlation functions, we consider the higher spin Heisenberg models. This case is more simple as the ground state of a higher spin *XXX* chain contains only strings of one kind and not a mixture of different types of strings as an arbitrary excited state. This problem is important also from another point of view as it can give some information about other integrable models including integrable quantum field theories.

In this paper we consider the correlation functions of the *XXX* higher spin chains. This model was first considered by Kulish *et al* in [8] where the notion of *fusion* was introduced. It was solved by means of the algebraic Bethe ansatz by Takhtajan [9] and independently by Babujian in [10]; the thermodynamics of this model was considered in [10]. The *XXZ* version of higher spin chains was introduced in [11] and solved in [12], but these models are not considered here for several reasons. The correlation functions of the *XXZ* spin 1 model in the anti-ferromagnetic regime were calculated in [13–15] using the vertex operator approach. Here we propose a different way of calculation of the correlation functions based on the algebraic Bethe ansatz.

As in [3] the first step of computation of the correlation functions is the solution of the *quantum inverse problem*. Such a solution for a very large class of quantum integrable models including the higher spin Heisenberg chains was recently obtained [5] in a form very similar to the spin $\frac{1}{2}$ case. Taken together with the results for the scalar products of the Bethe states [16–18] it permits obtaining a representation for the finite chain correlation functions for arbitrary spin. At this stage one should take the thermodynamic limit and, hence, introduce the string solutions. We illustrate this procedure using the simplest example of the spin 1 chain. We would like to underline that a similar procedure is possible also for higher spin models but it leads to more cumbersome calculations and we present only one form of the result for this case without detailed derivation.

The main difficulty which arises from the presence of bound states is the fact that the algebraic part becomes singular. This problem can be solved by choosing the integration contours in the multiple integral representations taking into account the sign of the finite size corrections to the string picture [19–21]. After this modification one can see that two parts appear again and the determinant part once again is defined uniquely by the ground state. Such a result is given in section 5.

The main result of this paper is that the mean values with respect to the states containing bound states can be calculated within the framework of our approach. It means in particular that some new tools introduced here can also be used to calculate the temperature-dependent correlation functions.

This paper is organized as follows. In section 2 we introduce the higher spin Heisenberg chains following the papers of Takhtajan [9] and Babujian [10]. The solution of the inverse problem for these models [5] is given in section 3. This solution is used to obtain representations for the correlation functions on a finite chain for arbitrary spin in section 4. We show, in particular, how to override the additional algebraic difficulties appearing in the higher spin case. The thermodynamic limit of the spin 1 chain is considered in the two last sections. We show how to deal with 2-strings in the thermodynamic limit using the simplest

example of one-point functions in section 5. This first example permits to elaborate some simple rules which deal with strings for general correlation functions (section 6).

2. *XXX* Heisenberg chain with arbitrary spin

In this section we introduce the *XXX* Heisenberg chains for arbitrary spin. Here, we follow in general the papers of Takhtajan [9] and Babujian [10].

Unlike the spin $\frac{1}{2}$ case we start directly from the L-operator and later construct the Hamiltonian from the transfer matrix. It is necessary to obtain an integrable generalization of the usual Heisenberg chain (a direct generalization of the spin $\frac{1}{2}$ *XXX* Hamiltonian is not integrable for higher spins). However, the L-operator can be obtained by direct generalization:

$$L_m(\lambda) = \frac{1}{\lambda - \mathrm{i}\left(s + \frac{1}{2}\right)} \begin{pmatrix} \lambda - \mathrm{i}\left(s_m^z + \frac{1}{2}\right) & -\mathrm{i}s_m^- \\ -\mathrm{i}s_m^+ & \lambda + \mathrm{i}\left(s_m^z + \frac{1}{2}\right) \end{pmatrix}. \tag{2.1}$$

One should note that for this L-operator the auxiliary space is two dimensional but the quantum space has $2s + 1$ dimensions. The matrices s^z, $s^{\pm}$ are the spin operators in the representation of spin s. This L-operator has the same intertwining relation with the rational 4×4 R matrix

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\lambda}{\lambda - \mathrm{i}} & \frac{-\mathrm{i}}{\lambda - \mathrm{i}} & 0 \\ 0 & \frac{-\mathrm{i}}{\lambda - \mathrm{i}} & \frac{\lambda}{\lambda - \mathrm{i}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

as in the spin $\frac{1}{2}$ case

$$R_{12}(\lambda - \mu)\ (L_m)_1(\lambda)\ (L_m)_2(\mu) = (L_m)_2(\mu)\ (L_m)_1(\lambda)\ R_{12}(\lambda - \mu). \tag{2.3}$$

As usual, indices 1 and 2 in L-operators denote two different auxiliary spaces.

The next step is the construction of the monodromy matrix for a spin chain of M sites

$$T(\lambda) = L_M(\lambda - \xi_M)L_{M-1}(\lambda - \xi_{M-1}) \cdots L_1(\lambda - \xi_1) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

with arbitrary inhomogeneity parameters ξ_j. For this monodromy matrix one can also write the 'commutation relation' with R-matrix

$$R_{12}(\lambda - \mu)\ T_1(\lambda)\ T_2(\mu) = T_2(\mu)\ T_1(\lambda)\ R_{12}(\lambda - \mu) \tag{2.4}$$

and it contains the commutation relation for the operators $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ acting in the quantum space.

Up to this point all the construction was the same as in the spin $\frac{1}{2}$ case but to construct local Hamiltonians from the monodromy matrix one should introduce some new concepts. The trace identities for the spin $\frac{1}{2}$ case were based on the fact that $L_n(0)$ is just a transposition operator in the tensor product of the auxiliary and quantum spaces. Now the quantum and auxiliary spaces have different numbers of dimensions. Hence it is necessary to construct a monodromy matrix with a $(2s + 1)$-dimensional auxiliary space. Such a construction was proposed by Kulish *et al* [8] and it is called *fusion*.

The fused L-operator $L_n^{(s)}(\lambda)$ can be constructed as a following projection on the symmetric subspace in the tensor product of $2s$ auxiliary spaces (this symmetric subspace has exactly $2s + 1$ dimensions) of the following product of local L-operators:

$$L_n^{(s)}(\lambda) = P^+_{a_1,\ldots a_n} L_{a_{2s},n}(\lambda + 2\mathrm{i}s - \mathrm{i}) \cdots L_{a_2,n}(\lambda + \mathrm{i}) L_{a_1,n}(\lambda) P^+_{a_1,\ldots a_n} \tag{2.5}$$

where indices a_j mark the corresponding auxiliary spaces and $P^+_{a_1,\dots a_n}$ is the symmetrizer (projector on the symmetric subspace). Thus we constructed the L-operator with $(2s+1)$-dimensional auxiliary space and satisfying the following property:

$$L^{(s)}_{a,n}(-\mathrm{i}s) = P_{a,n}$$

where $P_{a,n}$ is the transposition in the tensor product of the auxiliary space and local quantum space. This property is crucial for the construction of local Hamiltonians [10] and for the solution of the inverse problem [5]. The fused monodromy matrix is constructed as

$$T^{(s)}_a(\lambda) = L^{(s)}_M(\lambda-\xi_M)L^{(s)}_{M-1}(\lambda-\xi_{M-1})\cdots L^{(s)}_1(\lambda-\xi_1).$$

The fused transfer matrix, which is the trace of the monodromy matrix taken in the auxiliary space,

$$\tau^{(s)}(\lambda) = \mathrm{tr}_a T^{\boldsymbol{S}}_a(\lambda)$$

commutes not only with the transfer matrices for any value of parameter λ but also with the 'ordinary' transfer matrix $A(\lambda)+D(\lambda)$ (as it is a polynomial function of $A(\lambda)+D(\lambda)$):

$$\left[\tau^{(s)}(\lambda),\tau^{(s)}(\mu)\right] = \left[\tau^{(s)}(\lambda),A(\mu)+D(\mu)\right] = 0. \tag{2.6}$$

It means, in particular, that the Hamiltonians constructed from the fused transfer matrix in the homogeneous case $\xi_j = 0$ using the trace identities

$$H^{(s)} = \mathrm{const}\,\frac{\mathrm{d}}{\mathrm{d}\lambda}\tau^{(s)}(\lambda)\bigg|_{\lambda=-\mathrm{i}s} \tag{2.7}$$

also commute with the 'ordinary' transfer matrix and can be diagonalized by the *algebraic Bethe ansatz* [4] procedure. The Hamiltonians constructed by trace identities are local, translation invariant and can be written as polynomials of degree $2s$ of the local spin–spin interaction terms [9]

$$H^{(s)} = \sum_{m=1}^{M} Q_{2s}(\mathbf{s}_m\mathbf{s}_{m+1}) \tag{2.8}$$

$$Q_{2s}(x) = \sum_{j=1}^{2s}\left(\sum_{k=1}^{j}\frac{1}{k}\right)\prod_{\substack{l=0\\ l\neq j}}^{2s}\frac{x-x_l}{x_j-x_l} \tag{2.9}$$

where $\mathbf{s}_n = (s^x_n, s^y_n, s^z_n)$ are spin operators and $x_l = \frac{1}{2}[l(l+1)-2s(s+1)]$. The first example of such a Hamiltonian is the spin 1 case where

$$H^{(1)} = \frac{1}{4}\sum_{m=1}\mathbf{s}_m\mathbf{s}_{m+1} - (\mathbf{s}_m\mathbf{s}_{m+1})^2. \tag{2.10}$$

As we already mentioned, to diagonalize this Hamiltonian one can use the usual algebraic Bethe ansatz. We start from the ferromagnetic state $|0\rangle$ with all the spins up

$$s^+_n|0\rangle = 0 \qquad \forall n$$

which is an eigenstate of the Hamiltonians (2.7) and consider the action of the generalized creation operators $B(\lambda)$ on this state. It is easy to see that a state

$$B(\lambda_1)B(\lambda_2)\cdots B(\lambda_N)|0\rangle$$

is an eigenstate of the ordinary transfer matrix

$$(A(\mu)+D(\mu))B(\lambda_1)B(\lambda_2)\cdots B(\lambda_N)|0\rangle = \tau(\mu,\{\lambda_j\})B(\lambda_1)B(\lambda_2)\cdots B(\lambda_N)|0\rangle \tag{2.11}$$

and, hence, of the Hamiltonians if the parameters $\{\lambda_j\}$ satisfy the following *Bethe equations*

$$\varphi_j(\{\lambda\}) \equiv \left(\frac{\lambda + \mathrm{i}s}{\lambda - \mathrm{i}s}\right)^M \prod_{k=1}^{N} \frac{\lambda_j - \lambda_k - \mathrm{i}}{\lambda_j - \lambda_k + \mathrm{i}} = -1. \tag{2.12}$$

These Bethe states have exactly the same property as the Bethe states for the spin $\frac{1}{2}$ case; in particular, one can prove the Gaudin formula [16, 17] for their norms

$$\langle 0| \prod_{j=1}^{N} C(\lambda_j) \prod_{k=1}^{N} B(\lambda_k)|0\rangle = (-1)^N \prod_{j\neq k} \frac{\lambda_j - \lambda_k - \mathrm{i}}{\lambda_j - \lambda_k} \det_N \Phi'(\{\lambda\}) \tag{2.13}$$

$$\Phi'(\{\lambda\})_{ab} = -\mathrm{i}\frac{\partial}{\partial \lambda_b} \log \varphi_a(\{\lambda\})$$

and Slavnov formula [1, 18] for scalar products of a Bethe state $\prod_{k=1}^{N} B(\lambda_k)|0\rangle$ and a state $\langle 0| \prod_{j=1}^{N} C(\mu_j)$ with arbitrary set of parameters

$$\langle 0| \prod_{j=1}^{N} C(\mu_j) \prod_{k=1}^{N} B(\lambda_k)|0\rangle = \frac{\det_N T(\{\lambda, \mu\})}{\det_N V(\{\lambda, \mu\})} \tag{2.14}$$

$$T(\{\lambda, \mu\})_{ab} = \frac{\partial}{\partial \lambda_a} \tau(\mu_b, \{\lambda\}) \qquad V(\{\lambda, \mu\})_{ab} = \frac{1}{\lambda_a - \mu_b}$$

where $\tau(\mu_b, \{\lambda\})$ is the eigenvalue of the ordinary transfer matrix (2.11).

The thermodynamic limit for the higher spin Heisenberg model is slightly more complicated than in spin $\frac{1}{2}$ case. The ground state of the spin s *XXX* model in the thermodynamic limit can be constructed using the 'string' solutions of the Bethe equations. It was shown in [9] that for the spin s chain the ground state contains only strings of length $2s$. Such a string solution can be written as follows:

$$\lambda_k^a = \mu_k + \mathrm{i}\left(a - s - \frac{1}{2}\right)$$

where μ_k is real and called the string centre. The particularity of the ground state is the fact that it contains only strings of one particular length.

The density of string centres in the ground state can be obtained from the following integral equation similar to the Lieb equation in spin $\frac{1}{2}$ case:

$$\rho_{2s}(\lambda) + 2\sum_{a=1}^{2s-1} \int_{-\infty}^{\infty} \mathrm{d}\mu\, K_a(\lambda - \mu)\rho_{2s}(\mu) + \int_{-\infty}^{\infty} \mathrm{d}\mu\, K_{2s}(\lambda - \mu)\rho_{2s}(\mu)$$
$$= \sum_{k=1}^{2s} K_s\left(\lambda + \mathrm{i}\left(s + \frac{1}{2} - k\right)\right) \tag{2.15}$$

where the kernels $K_j(\lambda)$ are defined as

$$K_a(\lambda) = \frac{2a}{(\lambda + \mathrm{i}a)(\lambda - \mathrm{i}a)}. \tag{2.16}$$

The solution of this equation can be easily obtained

$$\rho_{2s}(\lambda) = \frac{1}{2\cosh(\pi\lambda)}. \tag{2.17}$$

It is a remarkable property of the spin chains that for any spin the density is the same as in the spin $\frac{1}{2}$ case, but here one should note that the ground states are very different, being constructed from different types of strings.

These results for thermodynamics of the spin chains of arbitrary spin obtained by Takhtajan [9] and Babujan in [10] will be used in the next sections for the calculation of the correlation functions.

3. Inverse problem

In this section we recall the solution of the inverse scattering problem for the spin s *XXX* chain. In other words we reconstruct the local spin operators in terms of the fused monodromy matrices. We follow the approach proposed by Maillet and Terras in [5].

To illustrate the results of [5] we start with the spin 1 chain. In this case the fused monodromy matrix has the following form:

$$T^{(2)}(\lambda) = \begin{pmatrix} A(\lambda+\mathrm{i})A(\lambda) & \frac{1}{\sqrt{2}}(A(\lambda+\mathrm{i})B(\lambda)+B(\lambda+\mathrm{i})A(\lambda)) & B(\lambda+\mathrm{i})B(\lambda) \\ \frac{1}{\sqrt{2}}(A(\lambda+\mathrm{i})C(\lambda) & \frac{1}{2}(A(\lambda+\mathrm{i})D(\lambda)+D(\lambda+\mathrm{i})A(\lambda) & \frac{1}{\sqrt{2}}(D(\lambda+\mathrm{i})B(\lambda) \\ +C(\lambda+\mathrm{i})A(\lambda)) & +B(\lambda+\mathrm{i})C(\lambda)+C(\lambda+\mathrm{i})B(\lambda)) & +B(\lambda+\mathrm{i})D(\lambda)) \\ C(\lambda+\mathrm{i})C(\lambda) & \frac{1}{\sqrt{2}}(D(\lambda+\mathrm{i})C(\lambda)+C(\lambda+\mathrm{i})D(\lambda)) & A(\lambda+\mathrm{i})A(\lambda) \end{pmatrix}.$$

This matrix can be used to construct the operators s_n^z and $s_n^{\pm}$, but here we are mostly interested in the reconstruction of the local elementary matrices

$$(E^{\alpha',\alpha})_{ab} = \delta_{\alpha' a}\delta_{\alpha b}.$$

The local operators $E_j^{\alpha'_j,\alpha_j}$ can be expressed in terms of the corresponding monodromy matrix elements:

$$E_m^{\alpha'_m,\alpha_m} = \prod_{j=1}^{m-1} \tau^{(2)}(\xi_j - \mathrm{i})\, T^{(2)}_{\alpha_m,\alpha'_m}(\xi_m - \mathrm{i}) \prod_{j=m+1}^{M} \tau^{(2)}(\xi_j - \mathrm{i}) \tag{3.1}$$

where $\tau^{(2)}(\lambda) = \mathrm{tr}_0\, T^{(2)}(\lambda)$ is the fused transfer matrix.

This result can be easily generalized for an arbitrary spin s. Now the fused monodromy matrix is a $(2s+1)\times(2s+1)$ matrix and its elements are again sums of different products of $2s$ fundamental monodromy matrix elements:

$$T^{(2s)}_{\alpha,\alpha'}(\lambda) = \frac{1}{\left(C_{2s}^{\alpha-1}C_{2s}^{\alpha'-1}\right)^{1/2}} \sum_{\substack{j_1+\cdots+j_{2s}-2s=\alpha-1 \\ k_1+\cdots+k_{2s}-2s=\alpha'-1}} T_{j_{2s}k_{2s}}(\lambda+2s\mathrm{i}-\mathrm{i})\cdots T_{j_2k_2}(\lambda+\mathrm{i})T_{j_1k_1}(\lambda) \tag{3.2}$$

where C_n^k are binomial coefficients. For example the corner matrix element $T^{(2s)}_{1,1}(\lambda)$ is just a product of $2s$ operators $A(\lambda - k\mathrm{i})$, $k = 0,\ldots,2s-1$.

To reconstruct the elementary local operators one should again dress these monodromy matrix elements by the corresponding fused transfer matrices:

$$E_m^{\alpha'_m,\alpha_m} = \prod_{j=1}^{m-1} \tau^{(2s)}(\xi_j - \mathrm{i}s)\, T^{(2s)}_{\alpha_m,\alpha'_m}(\xi_m - \mathrm{i}s) \prod_{j=m+1}^{M} \tau^{(2s)}(\xi_j - \mathrm{i}s). \tag{3.3}$$

The shifts of the inhomogeneity parameters are chosen in such a way that the eigenvalue of the fused transfer matrix taken in the points $\xi_j + \mathrm{i}s$ on a Bethe state is

$$\tau^{(2s)}(\xi_j - \mathrm{i}s)B(\lambda_1)\cdots B(\lambda_N)|0\rangle = \prod_{k=1}^{N} \frac{\lambda_k - \xi_j - \mathrm{i}s}{\lambda_k - \xi_j + \mathrm{i}s}\; B(\lambda_1)\cdots B(\lambda_N)|0\rangle. \tag{3.4}$$

This is the solution of the quantum inverse problem for the arbitrary spin Heisenberg chain. Now we can use these representations to calculate the correlation functions and to do it we should first of all understand how these complicated combinations of the fundamental monodromy matrix elements act on Bethe states.

4. Finite lattice correlation functions

As our ultimate goal is to calculate the mean values of products of local operators with respect to a Bethe state and, in particular to the ground state, following the same ideas as in [3] we should consider the action of the local operators on Bethe states. As the Bethe states are eigenstates of the fused transfer matrix we need only to consider the action of the elements of the fused monodromy matrix. The result of this action is the 'algebraic part' of the final expression for the correlation functions. We begin by considering the action of a single local operator on a Bethe state.

First of all we recall the action of the operators A, B and D (elements of the fundamental monodromy matrix) on a 'bra' Bethe state

$$\langle 0|\prod_{k=1}^{N} C(\lambda_k)\, A(\lambda_{N+1}) = \sum_{a'=1}^{N+1} a(\lambda_{a'}) \frac{\prod_{k=1}^{N}(\lambda_k - \lambda_{a'} - \mathrm{i})}{\prod_{\substack{k=1\\k\neq a'}}^{N+1}(\lambda_k - \lambda_{a'})} \langle 0|\prod_{\substack{k=1\\k\neq a'}}^{N+1} C(\lambda_k) \tag{4.1}$$

$$\langle 0|\prod_{k=1}^{N} C(\lambda_k)\, D(\lambda_{N+1}) = \sum_{a=1}^{N+1} d(\lambda_a) \frac{\prod_{k=1}^{N}(\lambda_a - \lambda_k - \mathrm{i})}{\prod_{\substack{k=1\\k\neq a}}^{N+1}(\lambda_a - \lambda_k)} \langle 0|\prod_{\substack{k=1\\k\neq a}}^{N+1} C(\lambda_k) \tag{4.2}$$

$$\begin{aligned}\langle 0|\prod_{k=1}^{N} C(\lambda_k)\, B(\lambda_{N+1}) &= \sum_{a=1}^{N+1} d(\lambda_a) \frac{\prod_{k=1}^{N}(\lambda_a - \lambda_k - \mathrm{i})}{\prod_{\substack{k=1\\k\neq a}}^{N+1}(\lambda_a - \lambda_k)} \\ &\times \sum_{\substack{a'=1\\a'\neq a}}^{N+1} \frac{a(\lambda_{a'})}{(\lambda_{N+1} - \lambda_{a'} - \mathrm{i})} \frac{\prod_{\substack{j=1\\j\neq a}}^{N+1}(\lambda_j - \lambda_{a'} - \mathrm{i})}{\prod_{\substack{j=1\\j\neq a,a'}}^{N+1}(\lambda_j - \lambda_{a'})} \langle 0|\prod_{\substack{k=1\\k\neq a,a'}}^{N+1} C(\lambda_k).\end{aligned} \tag{4.3}$$

It can be seen from these formulae that there are two different type of sums produced by the action of the monodromy matrix elements 'A-type' and 'D-type' (action of the operator B produce for example one 'A-type' sum and one 'D-type' sum). In our case the eigenvalues of the operators $A(\lambda)$ and $D(\lambda)$ in the ferromagnetic state are

$$a(\lambda) = 1 \qquad d(\lambda) = \prod_{j=1}^{M} \left(\frac{\lambda - \xi_j + \mathrm{i}s}{\lambda - \xi_j - \mathrm{i}s}\right).$$

One should note that in order to calculate the correlation functions we should act by the 'strings' of operators. Consider first the action of one local operator $E_m^{\alpha,\alpha}$. As shown in section 3 it can be written as a sum of ordered products of the fundamental monodromy matrix elements taken in the points $\xi - \mathrm{i}s, \xi - \mathrm{i}s + \mathrm{i}, \ldots, \xi + \mathrm{i}s - \mathrm{i}$. The monodromy matrix elements acting on the ground state produce the sums on the 'D-type' indices a_j and 'A-type' indices a'_j and the number of such sums is the same for all the products corresponding to a fixed local operator, namely $\alpha - 1$ 'D-type' sums and $2s - \alpha' + 1\rangle$ 'A-type' sums. Introducing the new notations $\lambda_{N+j} = \xi - \mathrm{i}(s - j + 1)$ we can just repeat the calculations for the spin $\frac{1}{2}$ [3],

however, taking into account that now $d(\lambda_{N+j}) \neq 0$ for $j > 1$:

$$\frac{\langle 0|\prod_{k=1}^{N} C(\lambda_k)\,(\tau^{(2s)}(\xi - \mathrm{i}s))^{-1}\prod_{j=1}^{2s} T_{\alpha_j,\alpha'_j}(\lambda_{N+j})\prod_{k=1}^{N} B(\lambda_k)|0\rangle}{\langle 0|\prod_{k=1}^{N} C(\lambda_k)\prod_{k=1}^{N} B(\lambda_k)|0\rangle} = \frac{1}{\prod_{k=1}^{2s-1}\mathrm{i}^k\,k!}\sum_{a_j,a'_j=1}^{N+2s} H_{\{a_j,a'_j\}}(\lambda_1,\ldots,\lambda_{N+2s})\frac{\det\Psi(\{a,a'\})}{\det\Phi'}$$

where the function H is defined as

$$H_{\{a_j,a'_j\}}(\{\lambda\}) = \frac{(-1)^{2s}}{\prod_{k>l}\left(\lambda_{b_k}-\lambda_{b_l}-\mathrm{i}\right)}\prod_{j=1}^{2s-\alpha+1}\left(\prod_{k=1}^{j-1}\left(\lambda_{a'_j}-\lambda_{N+k}+\mathrm{i}\right)\prod_{k=j+1}^{2s}\left(\lambda_{a'_j}-\lambda_{N+k}\right)\right) \times\prod_{j=1}^{\alpha-1} d\left(\lambda_{a_j}\right)\prod_{k=1}^{N}\frac{\lambda_{a_j}-\lambda_k-\mathrm{i}}{\lambda_{a_j}-\lambda_k+\mathrm{i}}\left(\prod_{k=1}^{j-1}\left(\lambda_{a_j}-\lambda_{N+k}-\mathrm{i}\right)\prod_{k=j+1}^{2s}\left(\lambda_{a_j}-\lambda_{N+k}\right)\right). \tag{4.4}$$

For the indices a_j, a'_j, b_j, etc we conserve the notation of the spin $\frac{1}{2}$ case:

$$\{b_1,\ldots,b_m\} = \{a'_{2s-\alpha+1},\ldots,a'_1,a_1,\ldots,a_{\alpha-1}\}.$$

The determinant in the denominator is the Gaudin determinant and the matrix in the numerator is also the Gaudin matrix with some replaced columns. We will consider this 'analytic part' in general later on in this section, and for the ground state for the spin 1 case in the last sections. One can easily obtain representations for the replaced columns from the scalar product formula. Here we will consider the 'algebraic part' of the expression (4.4). Taking into account values of the parameters λ_{N+k} we obtain

$$H_{\{a_j,a'_j\}}(\{\lambda\}) = \frac{(-1)^{\alpha-1}}{\prod_{k>l}\left(\lambda_{b_k}-\lambda_{b_l}-\mathrm{i}\right)}\prod_{j=1}^{\alpha-1} d\left(\lambda_{a_j}\right)\prod_{k=1}^{N}\frac{\lambda_{a_j}-\lambda_k-\mathrm{i}}{\lambda_{a_j}-\lambda_k+\mathrm{i}} \times\left(\prod_{k=1}^{2s-1}(\lambda_{a_j}-\xi+\mathrm{i}(s-k))\right)\prod_{j=1}^{2s-\alpha+1}\left(\prod_{k=1}^{j-1}\left(\lambda_{a'_j}-\xi+\mathrm{i}(s-k+2)\right)\right. \times\left.\prod_{k=j+1}^{2s}\left(\lambda_{a'_j}-\xi+\mathrm{i}(s-k+1)\right)\right). \tag{4.5}$$

Here one can see that the product corresponding to the 'D-type' parameters is the same for any element of the sum in (3.2). Moreover, from this result one can see that $a_j > N$ gives a non-zero contribution only if $a_j > N+1$ and there is $a_k = a_j - 1$, $k < j$. It leads to the conclusion that such a term should contain $d(\xi - \mathrm{i}s)$ which is zero. It means that as in spin $\frac{1}{2}$ case the summations over a_j should be taken only from 1 to N.

The product corresponding to the 'A-type' parameters is not the same for all the terms but taking the sum in (3.2) and symmetrizing over the permutations of the 'D-type' and 'A-type' parameters separately one can simplify it and finally obtain

$$f_\alpha(1,s) \equiv \frac{\langle 0|\prod_{k=1}^{N} C(\lambda_k)\,E_m^{\alpha,\alpha}\prod_{k=1}^{N} B(\lambda_k)|0\rangle}{\langle 0|\prod_{k=1}^{N} C(\lambda_k)\prod_{k=1}^{N} B(\lambda_k)|0\rangle} = (-1)^{\alpha-1}\frac{C_{2s}^{\alpha-1}}{\prod_{k=1}^{2s-1}\mathrm{i}^k\,k!}\sum_{a_j=1}^{N}\sum_{a'_j=1}^{N+2s}\mathbf{H}^{\alpha}_{\{a_j,a'_j\}}(\lambda_1,\ldots,\lambda_{N+2s})\frac{\det\Psi(\{a,a'\})}{\det\Phi'} \tag{4.6}$$

$$\mathbf{H}^{\alpha}_{\{a_j,a'_j\}}(\{\lambda\}) = \prod_{\substack{k,l=1\\k>l}}^{\alpha-1} \frac{\lambda_{a_k}-\lambda_{a_l}}{\left(\lambda_{a_k}-\lambda_{a_l}\right)^2+1} \prod_{\substack{k,l=1\\k>l}}^{2s-\alpha+1} \frac{\lambda_{a'_k}-\lambda_{a'_l}}{\left(\lambda_{a'_l}-\lambda_{a'_k}\right)^2+1}$$
$$\times \prod_{k=1}^{\alpha-1}\prod_{l=1}^{2s-\alpha+1} \frac{1}{\lambda_{a_k}-\lambda_{a'_l}-\mathrm{i}} \prod_{j=1}^{\alpha-1}\left(\prod_{k=1}^{2s-1}\left(\lambda_{a_j}-\xi+\mathrm{i}(s-k)\right)\right)$$
$$\times \prod_{j=1}^{2s-\alpha+1}\left(\prod_{k=1}^{2s-1}\left(\lambda_{a'_j}-\xi+\mathrm{i}(s-k+1)\right)\right). \tag{4.7}$$

One can see from this representation that terms with $a'_j > N$ produce non-zero contributions if $a'_j = N+2s$ or if there is $a'_k = a'_j + 1$. It means that the operators $C(\xi - \mathrm{i}(s-k))$ which appear in the scalar product after the action of the local operators should form a 'substring' without holes starting from $\xi - \mathrm{i}s$. For example, states like

$$\langle 0|C(\xi-\mathrm{i}s)C(\xi-\mathrm{i}(s-1))\cdots C(\xi-\mathrm{i}(s-k)) \prod_{\substack{b\leqslant N\\ b\neq a_j,a'_j}} C(\lambda_b)$$

produce non-zero contributions to the correlation functions but the contribution of states such as

$$\langle 0|C(\xi-\mathrm{i}s)C(\xi-\mathrm{i}(s-2)) \prod_{\substack{b\leqslant N\\ b\neq a_j,a'_j}} C(\lambda_b) \qquad \text{or} \qquad \langle 0|C(\xi-\mathrm{i}(s-1)) \prod_{\substack{b\leqslant N\\ b\neq a_j,a'_j}} C(\lambda_b)$$

is zero. This property is rather important as the matrix appearing in the scalar product is much simpler in this case.

Consider the determinant $\det\Psi$ appearing in (4.4) from the scalar product

$$\langle 0|C(\xi-\mathrm{i}s)C(\xi-\mathrm{i}(s-1))\cdots C(\xi-\mathrm{i}(s-k+1)) \prod_{b=k+1}^{N} C(\lambda_b) \prod_{a=1}^{N} B(\lambda_a)|0\rangle.$$

After extracting the normalization coefficients we obtain the following matrix:

$$\Psi_{ab} = \Phi'_{ab} \qquad b > k$$
$$\Psi_{a1} = \frac{1}{(\lambda_a-\xi+\mathrm{i}s)(\lambda_a-\xi+\mathrm{i}(s-1))} \qquad b = 1$$
$$\Psi_{ab} = \frac{1}{(\lambda_a-\xi+\mathrm{i}(s-b+1))(\lambda_a-\xi+\mathrm{i}(s-b))}$$
$$+ f_b \frac{1}{(\lambda_a-\xi+\mathrm{i}(s-b+2))(\lambda_b-\xi+\mathrm{i}(s-b+1))} \qquad 1 < b \leqslant k$$

where

$$f_b = d(\xi-\mathrm{i}(s-b+1)) \prod_{j=1}^{N} \frac{\lambda_j-\xi+\mathrm{i}(s-b+2)}{\lambda_j-\xi+\mathrm{i}(s-b)}.$$

The columns with $1 < b \leqslant k$ can be considered as sums of two columns and the second one does not contribute to the determinant as it is always linearly dependent on the columns with $a' < a$. Thus the matrix Ψ can be replaced by $\tilde{\Psi}$,

$$\tilde{\Psi}_{ab} = \Phi'_{ab} \qquad b > k$$
$$\tilde{\Psi}_{ab} = \frac{1}{(\lambda_a-\xi+\mathrm{i}(s-b+1))(\lambda_a-\xi+\mathrm{i}(s-b))} \equiv p'_b(\lambda_a-\xi) \qquad b \leqslant k.$$

A similar calculation can also be done for m-point functions leading to the following representation:

$$f_{\{\alpha,\alpha'\}}(m,s) \equiv \frac{\langle 0|\prod_{k=1}^{N} C(\lambda_k)\ \prod_{l=1}^{m} E_l^{\alpha'_l,\alpha_l}\ \prod_{k=1}^{N} B(\lambda_k)|0\rangle}{\langle 0|\prod_{k=1}^{N} C(\lambda_k)\ \prod_{k=1}^{N} B(\lambda_k)|0\rangle}$$
$$= (-1)^{\mathrm{P}(\alpha'_l-1)} \frac{\left(\prod_{l=1}^{m} C_{2s}^{\alpha_l-1} C_{2s}^{\alpha'_l-1}\right)^{1/2}}{\left(\prod_{k=1}^{2s-1} \mathrm{i}^k\, k!\right)^m \prod_{\substack{j,k=1\\ j>k}}^{m} \prod_{r=1}^{2s} \prod_{n=1}^{2r-1} (\xi_j - \xi_k - \mathrm{i}(r-n))}$$
$$\times \sum_{a_{jl}=1}^{N} \sum_{a'_{jl}=1}^{N+2sm} \mathbf{H}^{\alpha}_{\{a_{jl},a'_{jl}\}}(\{\lambda\}) \frac{\det\Psi(\{a,a'\})}{\det\Phi'}. \tag{4.8}$$

Here we introduced some new notations. In every site l there is a local operator $E_l^{\alpha'_l,\alpha_l}$ which produces sums over α_{l-1} 'D-type' indices a_{jl} and $2s+1-\alpha'_l$ 'A-type' indices a'_{jl}, and we define $\lambda_{N+2sl+k} = \xi_l - \mathrm{i}(s-k+1)$. We obtain the algebraic part:

$$\mathbf{H}^{\alpha,\alpha'}_{\{a_{jl},a'_{jl}\}}(\{\lambda\}) = \prod_{l=1}^{m} \prod_{k=1}^{2s-1} \left(\prod_{j=1}^{\alpha_l-1} \left(\lambda_{a_{jl}} - \xi_l + \mathrm{i}(s-k)\right) \prod_{j=1}^{2s+1-\alpha'_l} \left(\lambda_{a'_{jl}} - \xi_l + \mathrm{i}(s-k+1)\right) \right)$$
$$\times \prod_{n\leqslant l} G_{ln}^{DD}(\{a_{jl},a_{jn}\}) G_{ln}^{DA}(\{a_{jl},a'_{jn}\}) G_{ln}^{AD}(\{a'_{jl},a_{jn}\}) G_{ln}^{AA}(\{a'_{jl},a'_{jn}\}). \tag{4.9}$$

Where the 'two sites' contributions G_{ln} for two different sites l and n are

$$G_{ln}^{DD} = \frac{\prod_{j=1}^{\alpha_l-1} \left(\lambda_{a_{jl}} - \xi_n - \mathrm{i}s\right) \prod_{k=1}^{\alpha_n-1} \left(\lambda_{a_{kn}} - \xi_l + \mathrm{i}s\right)}{\prod_{j=1}^{\alpha_l-1} \prod_{k=1}^{\alpha_n-1} \left(\lambda_{a_{jl}} - \lambda_{a_{kn}} - \mathrm{i}\right)} \tag{4.10}$$

$$G_{ln}^{DA} = \frac{\prod_{j=1}^{\alpha_l-1} \left(\lambda_{a_{jl}} - \xi_n - \mathrm{i}s\right) \prod_{k=1}^{2s-\alpha_n+1} \left(\lambda_{a'_{kn}} - \xi_l - \mathrm{i}(s-1)\right)}{\prod_{j=1}^{\alpha_l-1} \prod_{k=1}^{2s-\alpha'_n+1} \left(\lambda_{a_{jl}} - \lambda_{a'_{kn}} - \mathrm{i}\right)} \tag{4.11}$$

$$G_{ln}^{AD} = \frac{\prod_{j=1}^{2s-\alpha'_l+1} \left(\lambda_{a'_{jl}} - \xi_n + \mathrm{i}(s+1)\right) \prod_{k=1}^{\alpha_n-1} \left(\lambda_{a_{kn}} - \xi_l + \mathrm{i}s\right)}{\prod_{j=1}^{2s-\alpha'_l+1} \prod_{k=1}^{\alpha_n-1} \left(\lambda_{a_{kn}} - \lambda_{a'_{jl}} - \mathrm{i}\right)} \tag{4.12}$$

$$G_{ln}^{AA} = \frac{\prod_{j=1}^{2s-\alpha'_l+1} \left(\lambda_{a'_{jl}} - \xi_n + \mathrm{i}(s+1)\right) \prod_{k=1}^{2s-\alpha_n+1} \left(\lambda_{a'_{kn}} - \xi_l - \mathrm{i}(s-1)\right)}{\prod_{j=1}^{2s-\alpha'_l+1} \prod_{k=1}^{2s-\alpha'_n+1} \left(\lambda_{a'_{kn}} - \lambda_{a'_{jl}} - \mathrm{i}\right)} \tag{4.13}$$

and the diagonal terms are given by

$$G_{ll}^{DD} G_{ll}^{AA} = \prod_{\substack{j,k=1\\ j>k}}^{\alpha_l-1} \frac{\lambda_{a_{jl}} - \lambda_{a_{kl}}}{\left(\lambda_{a_{jl}} - \lambda_{a_{kl}}\right)^2 + 1} \prod_{\substack{j,k=1\\ j>k}}^{2s-\alpha'_l+1} \frac{\lambda_{a'_{jl}} - \lambda_{a'_{kl}}}{\left(\lambda_{a'_{jl}} - \lambda_{a'_{kl}}\right)^2 + 1} \tag{4.14}$$

$$G_{ll}^{AD} G_{ll}^{DA} = \prod_{j=1}^{\alpha_l-1} \prod_{k=1}^{2s-\alpha'_l+1} \frac{1}{\lambda_{a_{jl}} - \lambda_{a'_{kl}} - \mathrm{i}}. \tag{4.15}$$

Here we obtained the algebraic part of the expression for an m-point correlation function for an arbitrary spin Heisenberg chain. Considering the determinant part we can easily

see using the same arguments as in the one-point case that the main proposition about 'the substrings'(the terms in the sum (4.8) with $a'_{jl} > N$ produce non-zero contributions if $a'_{jl} = N + 2sk\, k < l$ or if there is $a'_{kl} = a'_{jl} + 1$ with $k < j$) is also valid in the m-point case. It means that the determinants are always simple and contain only Gaudin columns and columns $p'_a(\lambda_b - \xi_k)$. More precisely, every term of the sum (4.8) contains a determinant which is obtained from the scalar product

$$\langle 0| \prod_{l=1}^{M} \prod_{j=1}^{k_l} C(\xi_l - \mathrm{i}(s - j + 1)) \prod_{b=k_1+\cdots+k_m+1}^{N} C(\lambda_b) \prod_{a=1}^{N} B(\lambda_a)|0\rangle$$

where $k_l = a'_{j_{\min} l} - 1$, the 'substring end' $a'_{j_{\min} l}$ being the minimal $a'_{jl} > N$ in the corresponding term. The corresponding matrix appearing in the sum (4.8) has the following columns:

$$\tilde{\Psi}_{ab} = \Phi'_{ab} \qquad b > \sum_{l=1}^{m} k_l$$

$$\tilde{\Psi}_{ab} = \frac{1}{(\lambda_a - \xi_l + \mathrm{i}(s - j + 1))(\lambda_a - \xi_l + \mathrm{i}(s - j))} = p'_j(\lambda_a - \xi_l) \qquad \sum_{r=1}^{l-1} k_r < b \leqslant \sum_{r=1}^{l} k_r$$

where $j = b - \sum_{r=1}^{l-1} k_r$.

Thus we have a representation for the correlation functions for a finite arbitrary spin Heisenberg chain. Being in some sense very similar to their spin $\frac{1}{2}$ counterparts these representations are rather complicated for big spins. For this reason in the next sections we will consider only the first generalization of the results of [3] which is the spin 1chain.

The next step of our approach is the thermodynamic limit for the ground state. On this stage the main difference with the spin $\frac{1}{2}$ case appears as the ground state is constructed of bound states (2s-strings). It produces some new difficulties which will be considered in the next section using the simplest example of the one-point functions.

5. One-point functions

To illustrate the last stage of the calculation, i.e. the introduction of the string solution of the Bethe equations for the ground state we begin with the simplest example, namely with the one-point functions $f_k(1)$ (corresponding to the diagonal elementary matrices E^{kk}).

Consider the simplest correlation function (one-point emptiness formation probability) of the spin 1 *XXX* chain in the homogeneous case:

$$f_3(1) = \langle \mathrm{vac}| \left(\tau_2^{(2)}\right)^{-1} (-\mathrm{i}) D(0) D(-\mathrm{i})|\mathrm{vac}\rangle. \tag{5.1}$$

Using the action of the operators D on the vacuum and the scalar product formula we easily obtain (there is no difference with the case spin $\frac{1}{2}$) for the finite chain:

$$f_3(1) = \mathrm{i} \sum_a \sum_{b \neq a} \frac{\lambda_a \lambda_b}{\lambda_a - \lambda_b - \mathrm{i}} \frac{\det \Psi}{\det \Phi'} \tag{5.2}$$

where Φ' is the Gaudin matrix:

$$\Phi'_{jk} = (MK(\lambda_j) - \sum_l K(\lambda_j - \lambda_l))\delta_{jk} + K(\lambda_j - \lambda_k) \tag{5.3}$$

with

$$K(\lambda) = \frac{2}{(\lambda + \mathrm{i})(\lambda - \mathrm{i})}$$

and the matrix Ψ is obtained from the scalar product:

$$\Psi_{jk} = \Phi'_{jk} \qquad j \neq a, b$$
$$\Psi_{ak} = \frac{1}{\lambda_k(\lambda_k - \mathrm{i})} \equiv p'_+(\lambda_k)$$
$$\Psi_{bk} = \frac{1}{\lambda_k(\lambda_k + \mathrm{i})} \equiv p'_-(\lambda_k).$$

Hence one can again divide one matrix by another and reduce it to a 2×2 matrix:

$$\left(\Phi'^{-1}\Psi\right)_{jk} = \delta_{jk} \qquad j \neq a, b$$
$$\left(\Phi'^{-1}\Psi\right)_{ak} = \phi_k^+$$
$$\left(\Phi'^{-1}\Psi\right)_{bk} = \phi_k^-$$

where $\phi_k^\pm$ are the solutions of the following systems of linear equations:

$$(MK(\lambda_k) - \sum_j K(\lambda_j - \lambda_k))\phi_k^\pm + \sum_j K(\lambda_j - \lambda_k))\phi_j^\pm = p'_\pm(\lambda_k). \tag{5.4}$$

Thus the one-point function (5.1) is given by

$$f_3(1) = \mathrm{i}\sum_a \sum_{b\neq a} \frac{\lambda_a\lambda_b}{\lambda_a - \lambda_b - \mathrm{i}} \det \begin{pmatrix} \phi_a^+ & \phi_b^+ \\ \phi_a^- & \phi_b^- \end{pmatrix}. \tag{5.5}$$

In the thermodynamic limit the ground state of the spin 1 *XXX* chain is built of 2-strings

$$M \to \infty: \quad \lambda_{2k-1} \to \mu_k + \frac{\mathrm{i}}{2} \qquad \lambda_{2k} \to \mu_k - \frac{\mathrm{i}}{2} \qquad \mathrm{Im}(\mu_\mathrm{k}) = 0.$$

To obtain the equations for the analytic part in the thermodynamic limit one should take into account the finite size corrections to this string picture as some terms in (5.4) become singular. To analyse the excited states with finite energy of the *XXX* spin $\frac{1}{2}$ one usually considers first the string limit and only then the thermodynamic limit, as these corrections are exponentially small. However, for the ground state of the spin 1 *XXX* model one cannot use this method in a rigorous way[2] as the corrections to the string picture calculated in [19–22] are of the order $\frac{1}{M}$:

$$\lambda_{2k-1} - \lambda_{2k} - \mathrm{i} = 2\mathrm{i}\frac{\alpha_k}{M} + o\left(\frac{1}{M}\right) \qquad \alpha > 0.$$

The correction α is always positive and it makes possible to rewrite the system of linear equations (5.4) as integral equations in the thermodynamic limit with a special choice of the integration contours near the singular point:

$$\begin{aligned} \varphi_1^\pm(\mu) + \int_{-\infty}^{\infty} \mathrm{d}\lambda\, K(\mu - \lambda)\varphi_1^\pm(\lambda) + \int_{-\infty}^{\infty} \mathrm{d}\lambda\, K(\mu - \lambda + \mathrm{i} + \mathrm{i}0)\varphi_2^\pm(\lambda) &= p'_\pm\left(\mu + \frac{\mathrm{i}}{2}\right) \\ \varphi_2^\pm(\mu) + \int_{-\infty}^{\infty} \mathrm{d}\lambda\, K(\mu - \lambda)\varphi_2^\pm(\lambda) + \int_{-\infty}^{\infty} \mathrm{d}\lambda\, K(\mu - \lambda - \mathrm{i} - \mathrm{i}0)\varphi_1^\pm(\lambda) &= p'_\pm\left(\mu - \frac{\mathrm{i}}{2}\right) \end{aligned} \tag{5.6}$$

where $\varphi_1^\pm(\mu_k) \equiv M\rho(\mu_k)\phi_{2k-1}^\pm$ and $\varphi_2^\pm(\mu_k) \equiv M\rho(\mu_k)\phi_{2k}^\pm$. Here we used the integral equation for density of strings in the ground state

$$\rho(\lambda) = \frac{1}{2\cosh(\pi\lambda)}.$$

[2] This method leads to the same result for the correlation functions as the one described later but in a more complicated way.

The solution of this system can be easily obtained,

$$\begin{aligned} \varphi_1^+(\lambda) &= \rho(\lambda) \qquad \varphi_2^+(\lambda) = 0 \\ \varphi_1^-(\lambda) &= 0 \qquad \varphi_2^-(\lambda) = \rho(\lambda). \end{aligned} \tag{5.7}$$

Now we can substitute this result to the expression for the one-point function (5.5), replacing sums by integral avoiding the singular point in the *same way* as in the integral equation (taking into account the sign of the finite size corrections) (5.6).

$$f_3(1) = \frac{\mathrm{i}}{4}\int_{-\infty}^{\infty}\mathrm{d}\lambda\int_{-\infty}^{\infty}\mathrm{d}\mu\frac{1}{\cosh(\pi\lambda)\cosh(\pi\mu)}\left(\frac{\left(\lambda+\frac{\mathrm{i}}{2}\right)\left(\mu-\frac{\mathrm{i}}{2}\right)}{\lambda-\mu+\mathrm{i}0}-\frac{\left(\lambda-\frac{\mathrm{i}}{2}\right)\left(\mu+\frac{\mathrm{i}}{2}\right)}{\lambda-\mu-2\mathrm{i}}\right). \tag{5.8}$$

One of these two integrals can be calculated as only the pole in the point $\mu = \lambda$ contributes. We finally obtain

$$f_3(1) = \frac{\pi}{2}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\frac{\lambda^2+\frac{1}{4}}{\cosh^2(\pi\lambda)} = \frac{1}{3}. \tag{5.9}$$

Of course this result can be obtained directly from the symmetry of the model, but this calculation illustrates well how to deal with strings in our method and it can be useful not only for the more general case of m-point functions which will be considered in the next section but also for the computation of more general correlation functions, depending, for example, on the temperature.

Two other one-point functions can be calculated in a rather similar way but here we should consider also the 'A-type' sums which contain more terms than 'D-type' sums considered in the previous example. We will show, using the simplest example of the one-point functions $f_2(1)$ and $f_1(1)$ that this problem can be solved exactly as in the spin $\frac{1}{2}$ case by moving the corresponding contour of integration.

Consider the function

$$f_2(1) = \frac{1}{2}\langle\mathrm{vac}|\tau_2^{-1}(-\mathrm{i})(A(0)D(-\mathrm{i}) + D(0)A(-\mathrm{i}) + C(0)B(-\mathrm{i}) + B(0)C(-\mathrm{i}))|\mathrm{vac}\rangle.$$

We easily obtain a finite lattice representation for this function (4.6)

$$f_2(1) = -2\mathrm{i}\sum_{a=1}^{N}\sum_{\substack{b=1\\b\neq a}}^{N}\frac{\lambda_a(\lambda_b+\mathrm{i})}{\lambda_a-\lambda_b-\mathrm{i}}\det\begin{pmatrix}\phi_a^+ & \phi_b^+\\ \phi_a^- & \phi_b^-\end{pmatrix} + 2\sum_a\frac{\lambda_a}{\lambda_a-\mathrm{i}}\phi_a^-. \tag{5.10}$$

One should mention that the sum over index a is a D-type sum and the sum over index b is a A-type sum and it contains one additional term (only one because of the substring limitation).

As in the previous case we can rewrite these sums as integrals in the thermodynamic limit:

$$\begin{aligned} f_2(1) = &-\frac{\mathrm{i}}{2}\int_{-\infty}^{\infty}\mathrm{d}\lambda\int_{-\infty}^{\infty}\mathrm{d}\mu\frac{1}{\cosh(\pi\lambda)\cosh(\pi\mu)}\left(\frac{\left(\lambda+\frac{\mathrm{i}}{2}\right)\left(\mu+\frac{\mathrm{i}}{2}\right)}{\lambda-\mu+\mathrm{i}0}-\frac{\left(\lambda-\frac{\mathrm{i}}{2}\right)\left(\mu+\frac{3\mathrm{i}}{2}\right)}{\lambda-\mu-2\mathrm{i}}\right)\\ &+\int_{-\infty}^{\infty}\mathrm{d}\lambda\frac{1}{\cosh(\pi\lambda)}\frac{\lambda-\frac{\mathrm{i}}{2}}{\lambda_a-3\frac{\mathrm{i}}{2}}. \end{aligned} \tag{5.11}$$

One should note that function $\rho(\lambda) = \frac{1}{2\cosh(\pi\lambda)}$ is the same as in the spin $\frac{1}{2}$ case and has a pole at $\lambda = -\frac{\mathrm{i}}{2}$ and its residue is

$$2\pi\mathrm{i}\,\mathrm{Res}\rho(\lambda)|_{\lambda=-\frac{\mathrm{i}}{2}} = -1.$$

It means in particular that shifting the contour of integration on the variable μ to the line parallel to the real axis with $\mathrm{Im}(\mu) = -1$ one crosses the pole of $\rho(\lambda)$ for the second double integral in (5.11) (for the first double integrals one can move the contour without crossing any poles) and the contribution of the pole is exactly similar to the single integral in (5.11). Finally we get

$$f_2(1) = \frac{\mathrm{i}}{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\mu \frac{1}{\cosh(\pi\lambda)\cosh(\pi\mu)} \left(\frac{\left(\lambda + \frac{\mathrm{i}}{2}\right)\left(\mu - \frac{\mathrm{i}}{2}\right)}{\lambda - \mu + \mathrm{i}} - \frac{\left(\lambda - \frac{\mathrm{i}}{2}\right)\left(\mu + \frac{\mathrm{i}}{2}\right)}{\lambda - \mu - \mathrm{i}} \right).$$

Here once again one can reduce this expression to a single integral:

$$f_2(1) = \frac{\pi}{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda \, \frac{\frac{1}{2} - 2\lambda^2}{\cosh^2(\pi\lambda)} = \frac{1}{3}. \tag{5.12}$$

Considering the last one-point function $f_1(1)$ we can obtain in a similar way that one should move both contours to obtain the same double integral representation as for $f_3(1)$:

$$f_1(1) = \frac{\pi}{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda \, \frac{\lambda^2 + \frac{1}{4}}{\cosh^2(\pi\lambda)} = \frac{1}{3}. \tag{5.13}$$

In this section we considered the simplest examples of the correlation functions; however, this simple example illustrates quite well the basic properties of the thermodynamic limits for the ground state constructed of the 2-strings. We have also shown that the A-type sums should be replaced by the integrations over shifted contours (as in spin $\frac{1}{2}$ case).

In the last section we show how this technique can be used for the general m-point functions for the spin 1 Heisenberg *XXX* chain. Their proofs are in general absolutely equivalent to the calculations in this section, but contain some very cumbersome formulae which we generally omit for the intermediate steps. Also, this method can be used for arbitrary spin and in the end we give one of the possible multiple integral representations for the correlation functions of the higher spin chains.

6. Correlation functions

In this section we generalize the results of the previous section for the general m-point equal-time correlation functions of the *XXX* chain spin 1 in the thermodynamic limit.

We calculate the following correlation functions or, more precisely, the elementary blocks which permit construction of any m-point correlation function:

$$f_{\{\alpha,\alpha'\}}(m) = \frac{\langle\psi_g| \prod_{j=1}^{m} E_j^{\alpha'_j,\alpha_j} |\psi_g\rangle}{\langle\psi_g|\psi_g\rangle} \tag{6.1}$$

where $E_j^{\alpha'_j,\alpha_j}$ are elementary local 3×3 matrices $E_{lk}^{\alpha',\alpha} = \delta_{l,\alpha'}\delta_{k,\alpha}$ and $|\psi_g\rangle$ is the ground state of the model (in the spin 1 case the number of quasi-particle N in the ground state is equal to the number of sites M).

For a finite spin 1 chain we obtained (4.8) that this correlation function can be represented as multiple sums

$$f_{\{\alpha,\alpha'\}}(m) = (-1)^{\mathrm{P}(\alpha'_l - 1)} \frac{\left(\prod_{l=1}^{m} C_2^{\alpha_l - 1} C_2^{\alpha'_l - 1}\right)^{1/2}}{\mathrm{i}^m \prod_{\substack{j,k=1 \\ j>k}}^{m} (\xi_j - \xi_k)^2((\xi_j - \xi_k)^2 + 1)} \times \sum_{\{a_j, a'_j\}} \mathbf{H}_{\{a,a'\}}(\{\lambda\}) \det_{2m} \tilde{S}(\{a, a'\}). \tag{6.2}$$

In the thermodynamic limit the solution of the Bethe equations corresponding to the ground state consists of 2-strings distributed with the following density:

$$\rho_{\rm tot}(\lambda) = \frac{1}{M}\sum_{n=1}^{M}\rho(\lambda-\xi_n).$$

As in the spin $\frac{1}{2}$ case it is convenient to introduce a set of indices b_k, which is the set of a_{jl}, a'_{jl} ordered in a special way:

$$\{b_1,\dots,b_{2m}\} = \{\{a'_{jm}\},\dots,\{a'_{j1}\},\{a_{j1}\},\dots,\{a_{jm}\}\}$$

with local subsets $\{a_{j1}\}$, $1\leqslant j\leqslant \alpha_l-1$ and $\{a'_{jl}\}$, $1\leqslant j\leqslant 3-\alpha'_l$, (these subsets can contain one, two or no elements).

Let us now consider the determinant part of the expression (4.8). In the thermodynamic limit we can divide the matrix in the numerator by the Gaudin matrix, or more precisely, we can calculate $\det(\Phi'^{-1}\Psi(\{a,a'\}))$. This can be written as a determinant of a $2m\times 2m$ matrix and the matrix elements are given by the inhomogeneous version of the integral equations (5.6) with $p_{\pm}(\mu-\xi_j\pm\frac{i}{2})$ in the right-hand side. Due to the translation invariance of this equations we obtain the same solution with a shift $\varphi^{\pm}_{1,2}(\mu-\xi_j)$. Finally we obtain

$$\begin{aligned}
&\det_M(\Phi'^{-1}\Psi(\{a,a'\})) = \det_{2m}\tilde S(\{a,a'\})\\
&\tilde S_{jk}(\{a,a'\}) = -\delta_{b_j-M,k} && b_j > M\\
&\tilde S_{j2k-1}(\{a,a'\}) = \frac{1+(-1)^{b_j}}{2}\,\frac{\rho\left(\lambda_{b_j}-\xi_k-\frac{\rm i}{2}\right)}{\rho_{\rm tot}\left(\lambda_{b_j}-\frac{\rm i}{2}\right)} && b_j\leqslant M\\
&\tilde S_{j2k}(\{a,a'\}) = \frac{1-(-1)^{b_j}}{2}\,\frac{\rho\left(\lambda_{b_j}-\xi_k+\frac{\rm i}{2}\right)}{\rho_{\rm tot}\left(\lambda_{b_j}+\frac{\rm i}{2}\right)} && b_j\leqslant M.
\end{aligned}\tag{6.3}$$

Now all the sums over b_l from 1 to M can be replaced by integrals taking into account that we obtain a sum of two integrals obtained from $b_l=2j$, $j=1,\dots,M/2$, and $b_l=2j-1$, $j=1,\dots,M/2$, dealing with the contours near the singularities of the algebraic part in the same way as in the previous section. Replacing sums by integrals we use the following rules and notations:

$$\sum_{\substack{b_l=2j-1\\ j=1}}^{M/2}\longrightarrow\int_{-\infty+\frac{\rm i}{2}+{\rm i}0}^{\infty+\frac{\rm i}{2}+{\rm i}0}{\rm d}\lambda_l\,\rho_{\rm tot}\left(\lambda_l-\frac{\rm i}{2}\right)\qquad \sum_{\substack{b_l=2j\\ j=1}}^{M/2}\longrightarrow\int_{-\infty-\frac{\rm i}{2}-{\rm i}0}^{\infty-\frac{\rm i}{2}-{\rm i}0}{\rm d}\lambda_l\,\rho_{\rm tot}\left(\lambda_l+\frac{\rm i}{2}\right)$$

$$\sum_{\substack{a_{kr}=2j-1\\ j=1}}^{M/2}\longrightarrow\int_{-\infty+\frac{\rm i}{2}+{\rm i}0}^{\infty+\frac{\rm i}{2}+{\rm i}0}{\rm d}\nu_{kr}\,\rho_{\rm tot}\left(\nu_{kr}-\frac{\rm i}{2}\right)\qquad \sum_{\substack{a_{kr}=2j\\ j=1}}^{M/2}\longrightarrow\int_{-\infty-\frac{\rm i}{2}-{\rm i}0}^{\infty-\frac{\rm i}{2}-{\rm i}0}{\rm d}\nu_{kr}\,\rho_{\rm tot}\left(\nu_{kr}+\frac{\rm i}{2}\right)$$

$$\sum_{\substack{a'_{kr}=2j-1\\ j=1}}^{M/2}\longrightarrow\int_{-\infty+\frac{\rm i}{2}+{\rm i}0}^{\infty+\frac{\rm i}{2}+{\rm i}0}{\rm d}\nu'_{kr}\,\rho_{\rm tot}\left(\nu'_{kr}-\frac{\rm i}{2}\right)\qquad \sum_{\substack{a'_{kr}=2j\\ j=1}}^{M/2}\longrightarrow\int_{-\infty-\frac{\rm i}{2}-{\rm i}0}^{\infty-\frac{\rm i}{2}-{\rm i}0}{\rm d}\nu'_{kr}\,\rho_{\rm tot}\left(\nu'_{kr}+\frac{\rm i}{2}\right).$$

In this notation the set $\{\lambda_l\}$ is the same set as $\{\nu_{kr},\nu'_{kr}\}$ but ordered in a special way:

$$\{\lambda_1,\dots,\lambda_{2m}\} = \{\{\nu'_{km}\},\dots,\{\nu'_{k1}\},\{\nu_{k1}\},\dots,\{\nu_{km}\}\}.$$

We will also use the following parameters:

$$\begin{aligned}
&\varepsilon_l = \frac{1}{2} && \text{if } \lambda_l=\nu_{kr} && (D\text{-type})\\
&\varepsilon_l = -\frac{1}{2} && \text{if } \lambda_l=\nu'_{kr} && (A\text{-type})
\end{aligned}$$

we can also associate to every λ_l the corresponding site number r_l if $\lambda_l = \nu_{kr_l}$ or $\lambda_l = \nu'_{kr_l}$. This notation is very useful in simplifying our formulae.

Replacing the sums by two integrals one should mention that for different contours for λ_j we obtain different functions in the determinant, namely $\varphi_1^{\pm}(\lambda_j - \xi_k - \frac{\mathrm{i}}{2})$ for the upper contour and $\varphi_2^{\pm}(\lambda_j - \xi_k + \frac{\mathrm{i}}{2})$ for the lower one. To simplify the formulae we introduce the following function:

$$\begin{aligned} \phi^{\pm}(\lambda) &= \varphi_1^{\pm}\left(\lambda - \frac{\mathrm{i}}{2}\right) \qquad && \mathrm{Im}(\lambda) > -\frac{1}{2} \\ \phi^{\pm}(\lambda) &= \varphi_2^{\pm}\left(\lambda + \frac{\mathrm{i}}{2}\right) \qquad && \mathrm{Im}(\lambda) < -\frac{1}{2}. \end{aligned} \tag{6.4}$$

(Note that we can deal with this function as with an analytic function if the integration contours do not cross the line $\mathrm{Im}(\lambda) = -\frac{1}{2}$).

We should now analyse the terms with $a'_{kr} > M$. As in the spin $\frac{1}{2}$ case they can be written as integrals around the corresponding poles of the determinant part in the points $\lambda_j = \xi_k$ (for the upper contour), $\lambda_j = \xi_k - i$ (for the lower contour). More precisely, we can replace the complete sum over $a'_{kr} > M$ by the following integral:

$$\sum_{a'_{kr}=2j} \longrightarrow \left(\int_{-\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0}^{\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0} + \sum_{l=1}^{m} \oint_{\Gamma'_l}\right) \mathrm{d}\nu'_{kr}\, \rho_{\mathrm{tot}}\left(\nu'_{kr} + \frac{\mathrm{i}}{2}\right)$$

$$\sum_{a'_{kr}=2j-1} \longrightarrow \left(\int_{-\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0}^{\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0} + \sum_{l=1}^{m} \oint_{\Gamma_l}\right) \mathrm{d}\nu'_{kr}\, \rho_{\mathrm{tot}}\left(\nu'_{kr} - \frac{\mathrm{i}}{2}\right).$$

where Γ_l and Γ'_l are small contours around the points ξ_l and $\xi_l - \mathrm{i}$. It means that as in the spin $\frac{1}{2}$ case the contribution of the poles can be absorbed into the integrals by moving all the contours for ν'_{kr} (A-type variables) down by i (to avoid crossing of some additional poles one should move first all the lower contours and then the upper ones).

Now we can write a multiple integral representation for the correlation functions of the spin 1 *XXX* chain:

$$\begin{aligned} f_{\{\alpha,\alpha'\}}(m) = \mathrm{i}^m \frac{\left(\prod_{l=1}^m C_2^{\alpha_l-1} C_2^{\alpha'_l-1}\right)^{1/2}}{\prod_{\substack{j,k=1\\ j>k}}^m (\xi_j-\xi_k)^2((\xi_j-\xi_k)^2+1)} \left(\int_{-\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0}^{\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0} + \int_{-\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0}^{\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0}\right) \mathrm{d}\lambda_1 \\ \cdots \left(\int_{-\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0}^{\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0} + \int_{-\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0}^{\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0}\right) \mathrm{d}\lambda_{2m}\, \mathbf{H}_{\{\alpha,\alpha'\}}(\{\lambda_l\})\, \det_{2m}\mathcal{S}(\{\lambda\}). \end{aligned} \tag{6.5}$$

The algebraic part for spin 1 can be written as

$$\mathbf{H}_{\{\alpha,\alpha'\}}(\{\lambda_l\}) = \prod_{l=1}^{2m}\prod_{k=1}^{m}(\lambda_l - \xi_k) \frac{\prod_{l=1}^{2m}\left(\prod_{k=1}^{r_l-1}(\lambda_l - \xi_k - 2\mathrm{i}\varepsilon_l)\prod_{k=r_l+1}^{m}(\lambda_l - \xi_k + 2\mathrm{i}\varepsilon_l)\right)}{\prod_{l>n}\left(\lambda_l - \lambda_n - \mathrm{i}(\varepsilon_l + \varepsilon_n)^2\right)} \tag{6.6}$$

and the $2m \times 2m$ matrix $\mathcal{S}$ is defined as

$$\begin{aligned} \mathcal{S}_{j,2k-1} &= \phi^-(\lambda_j - \xi_k) \\ \mathcal{S}_{j,2k} &= \phi^+(\lambda_j - \xi_k). \end{aligned} \tag{6.7}$$

This is one of many possible forms of the results which can be easily generalized to the higher spin cases. One should note that the correlation functions are written once again as a

multiple integral (or a sum of multiple integrals) and the integrals are taken over the solution of Bethe equations for the ground state (2-strings here). The expression under the integral can be once again separated into two distinctive parts: one defined by the choice of local operators (algebraic part) and the other by the ground state (determinant).

This result can be also rewritten in many different forms. First of all as for the one-point functions one can reduce the number of integration and obtain the result only as m integrals (this is a particularity of the spin 1 case and it cannot be done for the higher spins). First of all one should note that $S_{j,2k-1} = 0$ for upper integration contours and $S_{j,2k} = 0$ for lower ones (5.7). This means that the determinant has a block-diagonal structure and the integrals can be rewritten as the sum of integrals with m integrals over the upper contours and m integrals over lower ones:

$$f_{\{\alpha,\alpha'\}}(m) = \frac{\mathrm{i}^m \left(\prod_{l=1}^m C_2^{\alpha_l-1} C_2^{\alpha'_l-1}\right)^{1/2}}{\prod_{\substack{j,k=1\\ j>k}}^m (\xi_j-\xi_k)^2((\xi_j-\xi_k)^2+1)} \sum_{\{\lambda\}=\{\mu\}\cup\{\mu'\}} (-1)^{[\sigma]} \int_{-\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0}^{\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0} \mathrm{d}\mu_1 \cdots \int_{-\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0}^{\infty-\frac{\mathrm{i}}{2}-\mathrm{i}0} \mathrm{d}\mu_m \int_{-\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0}^{\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0} \mathrm{d}\mu'_1 \cdots \int_{-\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0}^{\infty+\frac{\mathrm{i}}{2}+\mathrm{i}0} \mathrm{d}\mu'_m \times \mathbf{H}_{\{\alpha,\alpha'\}}(\{\lambda_l\}) \det_m \mathcal{W}\left(\left\{\mu+\frac{\mathrm{i}}{2}\right\}\right) \mathcal{W}\left(\left\{\mu'-\frac{\mathrm{i}}{2}\right\}\right) \tag{6.8}$$

where the sum is taken over all possible partitions of the set $\{\lambda\}$ with $2m$ elements into two subsets with m elements and $[\sigma]$ is the sign of the following permutation:

$$\sigma(\{\lambda_1,\ldots,\lambda_{2m}\}) = \{\mu_1,\ldots,\mu_m,\mu'_1,\ldots,\mu'_m\}$$

and $\det \mathcal{W}(\{\mu'\})$ is the spin $\frac{1}{2}$ determinant

$$\mathcal{W}_{jk} = \rho(\mu_j - \xi_k).$$

This representation is also convenient as it contains only $m \times m$ matrices and only meromorphic functions. Now we can move the contours in this integral to obtain that only poles in $\lambda_j - \lambda_k + \mathrm{i}(\varepsilon_j + \varepsilon_K)$ contribute, which gives a representation of the result as a sum over all possible splitting of the set of $2m$ variables λ_l into m pairs which form strings $\lambda_l = \nu_j - \frac{\mathrm{i}}{2}$, $\lambda'_l = \nu_j + \frac{\mathrm{i}}{2}$ but every term contains only m-integrals over the string centres ν_j:

$$f_{\{\alpha,\alpha'\}}(m) = (2\pi)^m \frac{\left(\prod_{l=1}^m C_2^{\alpha_l-1} C_2^{\alpha'_l-1}\right)^{1/2}}{\prod_{\substack{j,k=1\\ j>k}}^m (\xi_j-\xi_k)^2((\xi_j-\xi_k)^2+1)} \int_{-\infty}^{\infty} \mathrm{d}\nu_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\nu_m \det_m{}^2 \mathcal{W}(\{\nu\}) \times \sum_{\{1,2,\ldots,2m\}=\cup_{j=1}^m \{l_j l'_j\}} G_{\{\alpha,\alpha'\}}(\{\nu\},\{\{l_j l'_j\}, j=1\ldots m\}). \tag{6.9}$$

The algebraic part here is however much more complicated than in the spin $\frac{1}{2}$ case. It is written as a sum of $(2m-1)!! \equiv 1 \times 3 \times \cdots \times (2m-1)$ terms, where every term is the corresponding residue of the general algebraic part $\mathbf{H}_{\{\alpha,\alpha'\}}$

$$G_{\{\alpha,\alpha'\}}(\{\nu\},\{\{l_j l'_j\}, j=1\ldots m\}) = \mathrm{Res}|_{\lambda_{l_1}=\lambda_{l'_1}+i(\varepsilon_{l_1}+\varepsilon_{l'_1})^2} \ldots \mathrm{Res}|_{\lambda_{l_m}=\lambda_{l'_m}+\mathrm{i}(\varepsilon_{l_m}+\varepsilon_{l'_m})^2} \mathbf{H}_{\{\alpha,\alpha'\}}(\{\lambda_l\}). \tag{6.10}$$

This is another possible form for the final result. It is rather particular as the correlation functions are represented as m-integrals as in the spin $\frac{1}{2}$ case and the analytic part is just the

square of the analytic part for spin $\frac{1}{2}$. It is also important to note that the homogeneous limit of this expression can be obtained exactly as in the spin $\frac{1}{2}$ case.

$$\lim_{\xi_j\to 0}\frac{\det_m^2 W(\{\mu\})}{\prod_{\substack{j,k=1\\ j>k}}^m(\xi_j-\xi_k)^2}=\det_m^2 W^{\hom}(\{\mu\})\qquad W^{\hom}_{jk}=\frac{1}{(k-1)!}\frac{\partial^{k-1}}{\partial\mu_j^{k-1}}\rho(\mu_j)\tag{6.11}$$

Once again the analytic part will be just the square of the corresponding spin $\frac{1}{2}$ algebraic part.

Thus we obtained several equivalent expressions for the correlation functions of the spin 1 Heisenberg *XXX* chain. It is interesting to note that this result looks quite different in comparison to the corresponding results obtained in [13, 14] (it is quite clear that for the same quantity one can write many different integral representations and sometimes it is rather difficult to prove that they are equivalent). The first (6.8) result has a form which is much simpler, even if we now have $2m$ integrals instead of m. Moreover similar results can be obtained for higher spins (4.8) in the thermodynamic limit, with more complicated contours (in some sense we integrate always *over the strings*), determinant of a $2ms\times 2ms$ matrix and corresponding general algebraic part:

$$\begin{aligned}f_{\{\alpha,\alpha'\}}(m,s)=&\frac{\left(\prod_{l=1}^m C_{2s}^{\alpha_l-1}C_{2s}^{\alpha'_l-1}\right)^{1/2}}{\left(\prod_{k=1}^{2s-1}\mathrm{i}^k\,k!\right)^m\prod_{\substack{j,k=1\\ j>k}}^m\prod_{r=1}^{2s}\prod_{n=1}^{2r-1}(\xi_j-\xi_k-\mathrm{i}(r-n))}\\&\times\int_{2s-\text{strings}}\mathrm{d}\lambda_1\ldots\int_{2s-\text{strings}}\mathrm{d}\lambda_{2sm}\;\mathbf{H}^{(s)}_{\{\alpha,\alpha'\}}(\lambda_1,\ldots,\lambda_{2sm})\det_{2sm}\mathcal{S}^{(s)}(\{\lambda\}).\end{aligned}\tag{6.12}$$

Here the intergals are taken over the strings with corresponding little shifts

$$\int_{2s-\text{strings}}\mathrm{d}\lambda\;f(\lambda)=\sum_{k=1}^{2s}\int_{-\infty-(\mathrm{i}+\mathrm{i}0)(2s-k-\frac{1}{2})}^{\infty-(\mathrm{i}+\mathrm{i}0)(2s-k-\frac{1}{2})}\mathrm{d}\lambda\;\;f(\lambda)$$

the algebraic part is

$$\begin{aligned}\mathbf{H}^{(s)}_{\{\alpha,\alpha'\}}(\{\lambda_l\})=&\left(\prod_{p=1}^{2s-1}\prod_{l=1}^{2sm}\prod_{k=1}^{m}(\lambda_l-\xi_k-\mathrm{i}(s-p))\right)\\&\times\frac{\prod_{l=1}^{2sm}\left(\prod_{k=1}^{r_l-1}(\lambda_l-\xi_k-2\mathrm{i}s\varepsilon_l)\prod_{k=r_l+1}^{m}(\lambda_l-\xi_k+2\mathrm{i}s\varepsilon_l)\right)}{\prod_{l>n}\left(\lambda_l-\lambda_n-\mathrm{i}(\varepsilon_l+\varepsilon_n)^2\right)}\end{aligned}\tag{6.13}$$

and the $2sm\times 2sm$ matrix $\mathcal{S}^{(s)}$ is defined as

$$\begin{aligned}\mathcal{S}_{j,2s(k-1)+l}=&\;\rho\left(\lambda_j-\xi_k+2s\mathrm{i}-\mathrm{i}l+\frac{\mathrm{i}}{2}\right)\qquad -2s+l-1<\mathrm{Im}(\lambda_j)<-2s+l\\&\;0\qquad\qquad\qquad\qquad\qquad\qquad\text{otherwise.}\end{aligned}\tag{6.14}$$

This representation can be proved in a very similar way to the spin 1 case (6.8).

In this paper we discussed the correlation functions of the higher spin *XXX* chains and have shown that even if the ground state contains bound states the correlation functions can be calculated. We hope to use these result to calculate the mean values of local operators with respect to any excited state for the spin $\frac{1}{2}$ Heisenberg chains and, hence, to obtain a representation for the finite temperature correlation functions.

Acknowledgments

I thank J-M Maillet, V Terras, N Slavnov, A Doikou and R Weston for many interesting discussions and comments. I am also grateful to N MacKay, E Corrigan, P Basillac and others members of the Department of Mathematics in York for the interest in my work. I also like to thank one of the referees for pointing out the results of the papers [21, 22]. This work was supported by UK EPSRC Grant GR/M 73231.

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